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General Transformation Theory of Lagrangian Mechanics and the Lagrange Group

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The general transformation theory of Lagrangian mechanics is revisited from a group-theoretic point of view. After considering the transformation of the Lagrangian function under local coordinate transformations in configuration spacetime, the general covariance of the formalism of Lagrange is discussed. Next, the group of Lagrange (for all n-dimensional Lagrangian systems) is introduced, and some important features of this group, as well as of its action on the set of Lagrangians, are briefly examined. Only finite local transformations of coordinates are considered here, and no variational transformation of the action is required in this study. Some miscellaneous examples of the formalism are included.

1. INTRODUCTION

It is well known that, from a practical point of view, the several formulations of mechanics usually do not materially decrease the difficulty of solving any given problem. For instance, in the Hamiltonian formulation one winds up practically with the same differential equations to be solved as are provided by the Lagrangian procedure. So, it seems that the advantages of the Lagrangian or the Hamiltonian formulations lie not so much in their use as calculational tools, but rather in the deeper insight they afford into the formal structure of mechanics.

Nevertheless, a given system can be described by more than one set of variables, and for each problem there may be one particular choice for which the variables may be more suitable since the solution is simpler. This fact sets the task of the *transformation theory of mechanics*, which studies the general conditions for the equal status accorded to different systems of variables. Thus, in Hamiltonian mechanics, this is the task of the theory of

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canonical transformations (see, for instance, Desloge, 1982, Vol. 2, pp. 755–764). Furthermore, transformation theory is an important subject, for it yields the basic tools for studying symmetries in mechanics.[3]

We will be interested here in dealing with the theory of transformations of Lagrangian mechanics. The focus of this theory is the existence of several *classes of equivalent Lagrangians* (i.e., "g-equivalent," "c-equivalent," or "s-equivalent" Lagrangians), which can be introduced in a very broad sense. Each class of Lagrangians plays a peculiar role within the general formalism. To handle them in a systematic fashion, one introduces first the concept of the *Lagrange group*, which is committed with the notion of "g-equivalence," as we shall see presently. This group is analogous to (and as important as) the *group of canonical transformations* of Hamiltonian mechanics. Essentially, the action of the elements of both groups is the same, notwithstanding the fact that they are not isomorphic; namely, they transform one mechanical system into another. The Lagrange group acts on the manifold of Lagrangians, the canonical group acts on the manifold of Hamiltonians, and both groups keep invariant the general mathematical features of the corresponding mechanical formalism.[4]

Most textbooks of analytical mechanics seldom consider the importance of the theory of transformations at the level of the Lagrangian formalism, as they always do for Hamiltonian mechanics and the Hamilton–Jacobi theory (Goldstein, 1980, Chapter 9, pp. 378–437, and Chapter 10, pp. 438–498), when considering canonical transformations. Apparently, the Lagrange group is the side of the transformation theory of mechanics that is least known to most physicists.

Lagrangian mechanics is one of the most important formalisms of physics, and the better we know it, the better shall be our understanding of many physical theories which are (or can be) embedded in the Lagrangian framework (Rosen, 1969). Hence, formal as it is, the general transformation theory of Lagrangian mechanics is an interesting subject by itself, which is as important as the theory of canonical transformations.

The organization of this paper is as follows. In Section 2 we recall some useful notions, and we review the gauge freedom of the Lagrangian formulation. In Section 3, the diffeomorphisms of point transformations are extended from the configuration space to the configration spacetime of the system, and the concept of a Lagrangian transformation is introduced.

[3]See Noether (1918). For an English translation of this fundamental paper, see Noether (1971).

[4]A unified treatment of symmetries in analytical mechanics has been recently proposed by Leubner and Marte (1985), in which an interesting generalized Noether theorem arises. Also a new unification approach is presented by Schafir (1988). These works point to the intimate relationship between both transformation groups, which, however, requires further study. In this sense, see also Kobe (1988).

Section 4 contains the discussion of the general covariance of the formalism under Lagrangian transformations. Section 5 is devoted to a brief discussion of the Lagrange group. We present three miscellaneous examples in Section 6, and finally, Section 7 contains some concluding remarks.

2. GAUGE TRANSFORMATIONS OF THE LAGRANGIAN

Let us first recall some useful notions. The equations of motion of an n-dimensional *Lagrangian system* are obtained from the Euler–Lagrange equations

$$\frac{\delta L}{\delta q^j} \equiv \frac{\partial L}{\partial q^j} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^j}\right) = 0 \tag{2.1}$$

of a given Lagrangian function $L(t, q, \dot{q}) = L(t; q^1, \ldots, q^n; \dot{q}^1, \ldots, \dot{q}^n)$. (It goes without saying that all Lagrangians considered in this paper correspond to systems with the same number of degrees of freedom.) Once a Lagrangian function has been found for the description of a system,[5] one obtains the Euler–Lagrange equations by applying Hamilton's variational principle to the action functional S defined by L.

An important class of Lagrangians (called "null Lagrangians") are those which have the property that *every* curve renders their action integrals stationary for all variations that vanish on the extremes. Their characterization is well known (Hill, 1951). Hence, the following concept is in order: Two Lagrangian functions $\tilde{L}(t, q, \dot{q})$ and $L(t, q, \dot{q})$ are said to be *g-equivalent* (i.e., gauge-equivalent) when there exist a function $G(t, q)$ and a constant K such that

$$\tilde{L}(t, q, \dot{q}) = KL(t, q, \dot{q}) + \dot{G}(t, q) \tag{2.2}$$

In such a case, performing variational derivatives on both members of equation (2.2), one gets

$$\frac{\delta L}{\delta q^j} = 0 \quad \Leftrightarrow \quad \frac{\delta \tilde{L}}{\delta q^j} \equiv K \frac{\delta L}{\delta q^j} = 0 \tag{2.3}$$

(since $\delta \dot{G}/\delta q^j \equiv 0$), which means that every solution of the old Euler–Lagrange equations is also a solution of the new Euler–Lagrange equations,

[5]This gives rise to the *inverse problem of the calculus of variations*, which consists in trying to find *all* Lagrangians which yield Euler–Lagrange equations that are equivalent to a given system of equations of motion. This problem was first solved for $n = 1$ by Darboux (1891, Vol. 3). The case $n = 2$ was treated much later by Douglas (1941). A large amount of noteworthy work devoted to this fundamental problem has been published lately; see, for instance, Sarlet (1982). For a recent approach, see Cariñena and Martínez (1989) and references therein.

and vice versa, so that g-equivalent Lagrangians provide the same equations of motion. This simply says that a Lagrangian function for a given system is only determined to within a *class* of g-equivalent Lagrangians defined by arbitrary *gauge transformations* of the form (2.2). (The interest of keeping the gauge scaling constant $K \neq 1$ will be apparent in the examples considered in this paper.)

Note that no transformation of variables participates in a gauge transformation of the Lagrangian. The new Lagrangian $\tilde{L}$ is just a *function* of the old Lagrangian L, such that both Lagrangians provide the same dynamical description of the system through the equations of motion. If one defines a new Lagrangian function $\tilde{L}(t, q, \dot{q}) = f(L) + \dot{G}(t, q)$, it can be shown that $f(L) = KL(t, q, \dot{q})$ (where K is a constant) yields the *most general* gauge transformation of L that is consistent with the implication stated in equation (2.3).

There is another kind of Lagrangian transformation (called "fouling" transformations) which satisfies also an implication analogous to equation (2.3). However, "fouling" transformations of L yield $\tilde{L}$ as a *functional* of L, while gauge transformations yield $\tilde{L}$ as a *function* of L (Curie and Saletan, 1966). All other transformations allowed in Lagrangian mechanics that are consistent with an implication *like* equation (2.3) are committed with transformations of variables. We shall discuss them in the sequel.

Although somehow trivial, gauge freedom is one of the most important features of Lagrangian mechanics. Indeed, the representative Lagrangian function that describes a given Lagrangian system is quite generally *not unique*, and g-equivalence entails the simplest instance of this fact.[6]

3. COORDINATE TRANSFORMATIONS IN CONFIGURATION SPACETIME

The configuration space $\{q\}$ of a Lagrangian system has the structure of a differentiable manifold, on which the group of n-dimensional (time-dependent) diffeomorphisms locally acts (Arnold, 1978). One easily proves the *general covariance* of the formalism of Lagrange under the action of this group.

The basic ideas and theorems of Lagrangian mechanics (even if formulated in terms of local coordinates) are invariant under a larger group of transformations which also affect time; furthermore, these transformations usually "mix" time with the generalized coordinates. For this reason, it is better if one considers the *configuration spacetime* $\{(t, q)\}$ as the fundamental

[6]There is another concept of "dynamically equivalent" Lagrangians (that is not considered here); this is the notion of *s-equivalent* Lagrangians, which was introduced by Hojman and Harleston (1981).

differentiable manifold of Lagrangian theory. The configuration spacetime of Lagrangian mechanics is not necessarily a metric space; it is just the space $\{t\}\times\{q\}$ of independent and dependent variables (Trümper, 1983, and references therein). Our aim is to discuss this subject, although rather briefly.

Let us consider a sufficiently smooth transformation of the variables $(t; q^1, \ldots, q^n)$ into a new set of variables $(T; Q^1, \ldots, Q^n)$:

$$\begin{aligned} T &= T(t, q) \\ Q^j &= Q^j(t, q) \end{aligned} \tag{3.1}$$

By "sufficiently smooth" one means that the functions T and Q^j are of continuity class C^μ $(\mu > 2)$ in some specified open connected region $R \subset \{(t, q)\}$, and globally invertible on R. (Usually one has C^∞, but this is not strictly necessary.) We shall write

$$\begin{aligned} t &= t(T, Q) \\ q^j &= q^j(T, Q) \end{aligned} \tag{3.2}$$

to denote the inverse transformation corresponding to (3.1). These are point transformations (i.e., diffeomorphisms) in configuration spacetime, and all admissible point transformations are assumed to meet these conditions. Henceforth, all our considerations have a *local* character, for we shall always assume that $(t, q) \in R$.

We here face a transformation [i.e., equation (3.1)] which we interpret either from a "passive" or from an "active" viewpoint. Although it usually matters little which intuitive point we adopt, at this stage we get a better development of these topics by presenting them under the scope of the "passive" point of view (which is also more akin with the theory of relativity). Thus, equation (3.1) will be thought of as a local *transformation of coordinates* in configuration spacetime (i.e., in this paper we interpret R as a *coordinate patch*). We next consider the action integral under this point of view.

In order to calculate a value for the functional S, one has to specify a curve $q^j = c^j(t)$; one then evaluates the action integral along the chosen curve, with $\dot{q}^j = dc^j(t)/dt$. In this fashion, given a transformation of coordinates, one writes $q^j = q^j(T, Q) = c^j[t(T, Q)]$, from which the expression $Q^j = C^j(T)$ for the curve follows in terms of the new coordinates (provided the conditions required by the implicit function theorem are satisfied). Hence we write, quite generally,

$$S = \int_{t_1}^{t_2} dt\, L(t, q, \dot{q}) = \int_{T_1}^{T_2} dT\, \hat{L}(T, Q, \mathring{Q}) = \hat{S} \tag{3.3}$$

where we define the *new Lagrangian* by

$$\dot{T}\hat{L}(T, Q, \mathring{Q}) = L(t, q, \dot{q}) \tag{3.4}$$

the *new generalized velocities* $\mathring{Q}^j$ corresponding to

$$\mathring{Q}^j = \frac{dQ^j}{dT} = \frac{\dot{Q}^j}{\dot{T}} \tag{3.5}$$

On the right-hand side of equation (3.3) we integrate along $Q^j = C^j(T)$ between the limits $T_1 = T[t_1, c(t_1)]$ and $T_2 = T[t_2, c(t_2)]$, since T is the new variable of integration. Note that equation (3.3) is valid for every chosen curve $q^j = c^j(t)$ whatsoever. In few words, equation (3.3) entails a simple change of variables in an integral, and therefore no question of symmetry for S is here involved.

Moreover, according to equation (3.4), one proves that gauge transformations of the Lagrangian are invariant under general coordinate transformations in configuration spacetime. This means that every local change of coordinates transforms whole g-classes of Lagrangians into new g-classes of Lagrangians. In this fashion, one justifies the following definition: Every local transformation of coordinates in configuration spacetime induces a new g-class of Lagrangian functions $\hat{L}$, which can be defined by

$$\dot{T}\hat{L}(T, Q, \mathring{Q}) = KL(t, q, \dot{q}) + \dot{G}(t, q) \tag{3.6}$$

where L is the old Lagrangian, G an arbitrary gauge function, K an arbitrary constant, and T is the new independent variable. This definition makes sense, because equation (3.6) differs from equation (3.4) by an arbitrary gauge transformation.[7] In the sequel we shall refer to equation (3.6) as a *Lagrangian transformation* induced by a local coordinate transformation in configuration spacetime.

A point worthy of mention is that one does not go too far within the Lagrangian theory of transformations if one adopts the "active" interpretation of diffeomorphisms, since there is no justifiable way for defining a *new* Lagrangian by means of an "active" mapping of events in configuration spacetime applied to the action integral. In fact, the "active" approach changes the *curve* along which S is defined; it does not change the *form* of the Lagrangian function. Thus, "active" mappings do not provide a criterion for defining a new Lagrangian, as coordinate transformations automatically do. [Of course, this remark does not preclude the use of the powerful variational (i.e., "active") approach to infinitesimal diffeomorphisms in mechanics, which is well suited for other important purposes.]

[7]Certainly, instead of equation (4.3), one now has $\hat{S} = KS + G_2 - G_1$, which corresponds to a gauge transformation of the action functional; see Levy-Leblond (1979).

4. GENERAL COVARIANCE OF THE LAGRANGIAN FORMALISM

We are ready to discuss the relationship between the new and the old Euler–Lagrange equations. First, we shall find the general law of covariance of the variational derivatives $\delta L/\delta q^j$ and $\delta \hat{L}/\delta Q^j$. Using equation (3.1), we now get

$$\begin{aligned}\frac{\partial \dot{T}}{\partial q^j}&=\frac{dT_j}{dt}=\dot{T}_j\\ \frac{\partial \dot{T}}{\partial \dot{q}^j}&=\frac{\partial T}{\partial q^j}=T_j\end{aligned} \tag{4.1}$$

and from equation (3.5) we have

$$\begin{aligned}\frac{\partial \mathring{Q}^k}{\partial q^j}&=\dot{T}^{-1}(Q_j^k-\dot{T}_j\mathring{Q}^k)\\ \frac{\partial \mathring{Q}^k}{\partial \dot{q}^j}&=\dot{T}^{-1}(Q_j^k-T_j\mathring{Q}^k)\end{aligned} \tag{4.2}$$

where $\dot{Q}_j^k=dQ_j^k/dt=\partial\dot{Q}^k/\partial q^j$. Thus, by means of equation (4.6) (taking $G=0$, which is enough for this purpose), we obtain

$$K\frac{\partial L}{\partial q^j}=\dot{T}_j\hat{L}+\dot{T}\left(T_j\frac{\partial \hat{L}}{\partial T}+Q_j^k\frac{\partial \hat{L}}{\partial Q^k}\right)+(\dot{Q}_j^k-\dot{T}_j\mathring{Q}^k)\frac{\partial \hat{L}}{\partial \mathring{Q}^k} \tag{4.3}$$

$$K\frac{\partial L}{\partial \dot{q}^j}=T_j\hat{L}+(Q_j^k-T_j\mathring{Q}^k)\frac{\partial \hat{L}}{\partial \mathring{Q}^k} \tag{4.4}$$

and hence

$$\begin{aligned}K\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^j}\right)&=\dot{T}_j\hat{L}+T_j\left(\dot{T}\frac{\partial \hat{L}}{\partial T}+\dot{Q}^k\frac{\partial \hat{L}}{\partial Q^k}\right)+\dot{T}(Q_j^k-T_j\mathring{Q}^k)\frac{d}{dT}\left(\frac{\partial \hat{L}}{\partial \mathring{Q}^k}\right)\\ &\quad+(\dot{Q}_j^k-\dot{T}_j\mathring{Q}^k)\frac{\partial \hat{L}}{\partial \mathring{Q}^k}\end{aligned} \tag{4.5}$$

follows. Therefore, equations (4.3) and (4.5) yield the desired covariance law.[8]

Thus, one proves the following theorem: The covariance law for the variational derivatives of two Lagrangian functions, which are related by a Lagrangian transformation in configuration spacetime, reads

$$\frac{\delta L}{\delta q^j}=K^{-1}\left(\dot{T}\frac{\partial Q^k}{\partial q^j}-\dot{Q}^k\frac{\partial T}{\partial q^j}\right)\frac{\delta \hat{L}}{\delta Q^k} \tag{4.6}$$

[8]A very thoughtful discussion of this subject (in the realm of classical field theory) can be found in Olver (1986, Chapter 4, pp. 246–286).

Note that this law can be inverted, so that both variational derivatives $\delta L/\delta q^j$ and $\delta\hat{L}/\delta Q^j$ stand on the same footing. This shows that the variational derivatives of the Lagrangian behave as a *geometric object* (of a very peculiar kind) under general coordinate transformations in configuration spacetime (see, for instance, Yano, 1955, p. 18).

Our next task is to ascertain the invariance of the Hamilton principle under general $(n+1)$-dimensional diffeomorphisms. Thus, we need to prove that if one directly applies the variational principle to the action S, then the *induced variation* on $\hat{S}$ [cf. equation (3.3)] corresponds to an application of the *same* principle directly on $\hat{S}$, and conversely. This is a consequence of the *general variational formula* for δS [due to generalized variations of the form $t \to t + \delta t(t, q)$ and $q^j \to q^j + \delta q^j(t, q)$], namely

$$\delta S = \int_{t_1}^{t_2} dt \frac{\delta L}{\delta q^j} (\delta q^j - \dot{q}^j \delta t) + \left[\frac{\delta L}{\delta \dot{q}^j} \delta q^j - \left(\frac{\delta L}{\delta \dot{q}^j} \dot{q}^j - L \right) \delta t \right]_{t_1}^{t_2} \tag{4.7}$$

from which the Hamilton principle follows as a special application (Mercier, 1963, pp. 12–14). We leave the details of the proof to the reader.

This finishes the "consistency control" of the formalism of Lagrangian transformations. Hence, one has that *Lagrangian mechanics is a general covariant theory under all diffeomorphisms which act as local coordinate transformations in configuration spacetime.*

The fact that the new coordinates $\{(T, Q)\}$ are "moving" relative to the old coordinates $\{(t, q)\}$ is tantamount to a substantial change in the dynamical description of the system. Indeed, from (4.6) it follows that

$$\frac{\delta L}{\delta q^j} \doteq 0 \quad \Leftrightarrow \quad \frac{\delta \hat{L}}{\delta Q^j} \equiv \frac{\partial \hat{L}}{\partial Q^j} - \frac{d}{dT}\left(\frac{\partial \hat{L}}{\partial \mathring{Q}^j} \right) \stackrel{\circ}{=} 0 \tag{4.8}$$

where the new equations of motion in general differ from the old equations of motion.[9] This means that whenever a trajectory $q^j = c^j(t)$ is a solution to the old Euler–Lagrange equations, the transformed trajectory $Q^j = C^j(T)$ will automatically satisfy the new equations of motion. Hence, the same change of coordinates that transforms one Lagrangian into another also transforms all the configuration worldlines of the system.

The local nature of diffeomorphisms must be stressed at this point. For instance, the harmonic oscillator and the free particle are very different dynamical systems, and there is no way that they can be mapped *globally* into each other by means of a Lagrangian transformation; however, pieces of them can be mapped into each other in local connected regions of

[9]The equality $A \doteq B$ means $A = B$ holds on the *physical wordlines* of the system, while it is not necessarily valid everywhere. (Of course, $A \stackrel{\circ}{=} B$ has the same meaning, *mutatis mutandi.*) Cf. Candotti *et al.* (1972).

spacetime, by means of local coordinate transformations that do not include any singular points. At first sight this seems to be almost intuitively "evident." Nevertheless, this is *not* the case for *any* two conceivable n-dimensional Lagrangian systems, which are characterized by two *given* Lagrangian functions L and L'. Contrary to the naïve intuitive "guess," the fact that there exists a local Lagrangian transformation that changes $L(t, q, \dot{q})$ into $\hat{L}(T, Q, \mathring{Q})$, far from being the rule, corresponds to the exception. In fact, it is not true that all Lagrangian functions are equivalent within a Lagrangian transformation, otherwise these transformations would be trivial. When there exists a Lagrangian transformation that maps one Lagrangian function into another one says that they are *c-equivalent* (i.e., curve-equivalent) Lagrangians. This feature makes Lagrangian transformations an important tool of analytical mechanics. We shall consider this subject in a forthcoming article. To this end, however, we need to know first some rather simple group properties of Lagrangian transformations.

5. THE LAGRANGE GROUP FOR *n*-DIMENSIONAL SYSTEMS

It is very useful to introduce a short-hand notation in order to discuss the group properties of diffeomorphisms and gauge transformations in Lagrangian mechanics. Henceforth we shall write D to denote a general diffeomorphism in configuration spacetime, as defined in equation (3.1), and D^{-1} to denote its inverse (3.2). Of course, the symbolic expression $D_{21} = D_2 D_1$ denotes the composite diffeomorphism

$$\begin{aligned} T &= T_2[T_1(t, q), Q_1(t, q)] \\ Q^j &= Q_2^j[T_1(t, q), Q_1(t, q)] \end{aligned} \tag{5.1}$$

In particular, we write I for the identity transformation: $T = t$ and $Q^j = q^j$.

If one performs two successive Lagrangian transformations (3.6), one gets a Lagrangian transformation given by

$$\dot{T}\hat{L}(T, Q, \mathring{Q}) = K_{21} L(t, q, \dot{q}) + \dot{G}_{21}(t, q) \tag{5.2}$$

where T and Q are given in (5.1), $K_{21} = K_2 K_1$ and G_{21} is defined by

$$G_{21}(t, q) = K_2 G_1(t, q) + G_2[T_1(t, q), Q_1(t, q)] \tag{5.3}$$

For the inversion of a Lagrangian transformation, one has

$$\mathring{t} L(t, q, \dot{q}) = K^{-1} \hat{L}(T, Q, \mathring{Q}) - K^{-1} \mathring{G}(t, q)$$

So, it is helpful to define G as a scalar function:

$$\hat{G}(T, Q) = G[t(T, Q), q(T, Q)] = G(t, q) \tag{5.4}$$

Accordingly, let us write the symbolic equation

$$\hat{L} = (D, K, G) L \tag{5.5}$$

to briefly denote a general Lagrangian transformation of L under D, according to the law stated in equation (3.6). In this manner, we can write

$$\tilde{L}=(I, K, G)L \tag{5.6}$$

as a symbol for the gauge transformation (2.3) of L generated by K and G. Thus, we get a very handy notation. For instance, one has the following composition law for these symbols:

$$(D_{21}, K_{21}, G_{21})=(D_2, K_2, G_2)(D_1, K_1, G_1)=(D_2D_1, K_2K_1, K_2G_1+G_2) \tag{5.7}$$

and the inversion law reads $(D, K, G)^{-1}=(D^{-1}, K^{-1}, -K^{-1}\hat{G})$, which meanings are sufficiently clear. Of course, one has $L=(I, 1, 0)L$, so that $(I, 1, 0)$ symbolizes the identity in the present notation. [As an application of this symbolism, it can be shown that Lagrangian transformations (D, K, G) and gauge transformations (I, K, G) obey the following commutation rule:

$$(D, K_2, G_2)(I, K_1, G_1)=(I, K_1, K_2\hat{G}_1-K_1\hat{G}_2+\hat{G}_2)(D, K_2, G_2) \tag{5.8}$$

By the way, this shows that each Lagrangian transformation (D, K, G) maps whole g-classes into new g-classes of Lagrangians.]

Since diffeomorphisms are always associative, i.e., $D_3(D_2D_1)=(D_3D_2)D_1$, the associative property of the product law (5.7) follows in a straightforward manner. Hence, the set of all Lagrangian transformations constitutes a group under this particular law of combination. We call this group the *Lagrange group* for n-dimensional Lagrangian systems, and we shall denote it by $L_{(n)}=\{(D, K, G)\}$. This is the *most general group of point transformations acting on the set* $\{L\}$ (of all representative Lagrangians for n-dimensional systems) *that keeps invariant the Lagrangian formalism of mechanics.*

For all the elements of $L_{(n)}$ one has

$$(D, 1, 0)(I, K, G)=(I, K, \hat{G})(D, 1, 0)=(D, K, G) \tag{5.9}$$

Furthermore, the Lagrange group $L_{(n)}$ is the *direct product* of the group $D_{(n+1)}=\{D\}$ of all $(n+1)$-dimensional local diffeomorphisms, and of the group $G_{(n)}=\{(I, K, G)\}$ of all gauge transformations. Thus, we write

$$L_{(n)}=D_{(n+1)}\otimes G_{(n)} \tag{5.10}$$

The group $D_{(n+1)}$ is isomorphic with the subgroup $\{(D, 1, 0)\}$ of $L_{(n)}$. Let us also observe that $G_{(n)}$ is a non-Abelian group (i.e., when $K\neq 1$), while $\{(I, 1, G)\}$ is an Abelian subgroup of $G_{(n)}$.

6. SOME MISCELLANEOUS EXAMPLES

In this section we present three interesting instances of the previous formalism. For the sake of briefness, we describe this matter in a rather sketchy manner.

6.1. Gauge Transformations and the Lorentz Force

The complete force on a point charge e moving in an electromagnetic field is

$$\mathbf{F}=e(\mathbf{E}+\dot{\mathbf{x}}\times\mathbf{B}) \tag{6.1}$$

where $\mathbf{E}$ and $\mathbf{B}$ are obtained from the potentials ϕ and $\mathbf{A}$, i.e.,

$$\begin{aligned}\mathbf{E}&=-\nabla\phi-\frac{\partial\mathbf{A}}{\partial t}\\ \mathbf{B}&=\nabla\times\mathbf{A}\end{aligned} \tag{6.2}$$

As is well known, if one introduces the generalized potential

$$U(t,\mathbf{x},\dot{\mathbf{x}})=e\left(\phi(t,\mathbf{x})-\frac{e}{c}\dot{\mathbf{x}}\cdot\mathbf{A}(t,\mathbf{x})\right) \tag{6.3}$$

and the (nonrelativistic) Lagrangian function

$$L(t,\mathbf{x},\dot{\mathbf{x}})=\tfrac{1}{2}m\dot{\mathbf{x}}^2-U(t,\mathbf{x},\dot{\mathbf{x}}) \tag{6.4}$$

where m is the mass of the charged particle, then the Euler–Lagrange equations yield

$$\frac{\delta L}{\delta\mathbf{x}}=-m\ddot{\mathbf{x}}-\frac{\delta U}{\delta\mathbf{x}}=-m\ddot{\mathbf{x}}+e(\mathbf{E}+\dot{\mathbf{x}}\times\mathbf{B})=0 \tag{6.5}$$

The electromagnetic potentials $(\phi,\mathbf{A})$ associated with a given electromagnetic field $(\mathbf{E},\mathbf{B})$ [cf. equations (6.2)] are defined to within a gauge transformation, namely

$$\begin{aligned}\phi\to\tilde{\phi}&=\phi+\frac{\partial\psi}{\partial t}\\ \mathbf{A}\to\tilde{\mathbf{A}}&=\mathbf{A}-\nabla\psi\end{aligned} \tag{6.6}$$

where $\psi(t,\mathbf{x})$ is an arbitrary scalar gauge field. Furthermore, equation (6.5) tells us that the acceleration $\ddot{\mathbf{x}}$ of a test particle in a given field depends only on its specific charge e/m. Hence, the allowed worldlines are invariant under the following change of scale:

$$\begin{aligned}m\to\tilde{m}&=Km\\ e\to\tilde{e}&=Ke\end{aligned} \tag{6.7}$$

Now, the transformations (6.6) and (6.7) induce the following gauge transformation of the Lagrangian defined in equation (6.4):

$$\tilde{L}(t, \mathbf{x}, \dot{\mathbf{x}}) = KL(t, \mathbf{x}, \dot{\mathbf{x}}) + \frac{d}{dt}[e\psi(t, \mathbf{x})] \tag{6.8}$$

Indeed, it is well known that these standard gauge transformations of electrodynamics yield one of the most important instances of equation (2.2).

6.2. The Free Particle and the Simple Harmonic Oscillator

It is well known that the local coordinate transformation (Arnold, 1988, p. 44)

$$T = \tan \omega t, \qquad Q = q \sec \omega t \tag{6.9}$$

reduces the equation of motion $\ddot{q} + \omega^2 q = 0$ into the free particle equation $\ddot{Q} = 0$. In fact, if we consider the Lagrangian function $\hat{L}(\mathring{q}) = \frac{1}{2}\mathring{q}^2$ under this diffeomorphism, after some manipulations we obtain

$$\tfrac{1}{2}\dot{T}\mathring{Q}^2 = \frac{1}{2\omega}(\dot{q}^2 - \omega^2 q^2) + \frac{d}{dt}(\tfrac{1}{2}q^2 \tan \omega t) \tag{6.10}$$

This is a Lagrangian transformation of the Lagrangian function $L(q, \dot{q}) = \frac{1}{2}(\dot{q}^2 - \omega^2 q^2)$, in the sense of equation (3.6), where we recognize $K = \omega^{-1}$ and

$$G(t, q) = \tfrac{1}{2}q^2 \tan \omega t \tag{6.11}$$

Hence, we see that the interest of the generalization (3.6) of equation (3.4) is not merely academic; as a matter of fact, it applies in many important examples of Lagrangian mechanics.

6.3. The One-Dimensional Kepler System

We finally present a counterexample, which shows that not all Lagrangian systems are c-equivalent, and therefore the formalism of Lagrangian transformations is not a trivial subject.

Let us consider the one-dimensional system defined by the Lagrangian $L(q, \dot{q}) = \frac{1}{2}\dot{q}^2 + (k/q)$, where k is a positive constant and $q > 0$. Assume that there exists a Lagrangian transformation (D, K, G) such that

$$\tfrac{1}{2}\dot{T}\mathring{Q}^2 = K\left(\tfrac{1}{2}\dot{q}^2 + \frac{k}{q}\right) + \dot{Q} \tag{6.12}$$

so that this system would be c-equivalent to a free particle system. More explicitly, this equation reads

$$KT_q\dot{q}^3+(KT_t+2T_qG_q-Q_q^2)\dot{q}^2+\left(2K\frac{k}{q}T_q+2T_tG_q+2T_qG_t-2Q_tQ_q\right)\dot{q}$$

$$+\left(2K\frac{k}{q}T_t+2T_tG_t-Q_t^2\right)=0 \tag{6.13}$$

where we have written $T_t=\partial T/\partial t$, $T_q=\partial T/\partial q$, and so on. Since equation (6.13) must hold identically for all values of $\dot{q}$, and $K\neq 0$, we get $T_q=0$ [i.e., $T=T(t)$], and therefore equation (6.13) requires

$$K\dot{T}-Q_q^2=0 \tag{6.14a}$$

$$\dot{T}G_q-Q_tQ_q=0 \tag{6.14b}$$

$$2\left(K\frac{k}{q}+G_t\right)\dot{T}-Q_t^2=0 \tag{6.14c}$$

From equation (6.14a) we obtain

$$Q(t,q)=\psi(t)q+\phi(t) \tag{6.15}$$

where $\psi(t)=(K\dot{T})^{1/2}$ and $\phi(t)$ remains arbitrary. Then, substitution of (6.15) into (6.14b) gives us

$$G(t,q)=\frac{\psi\dot{\psi}}{2\dot{T}}q^2+\frac{\dot{\phi}\psi}{\dot{T}}q+\mu(t) \tag{6.16}$$

with $\mu(t)$ a new arbitrary function.

However, substitution of the previous results into equation (6.14c), after some manipulations, yields

$$\left\{\frac{d}{dt}\left(\frac{\psi\dot{\psi}}{2\dot{T}}\right)-\frac{\dot{\psi}^2}{\dot{T}}\right\}q^3+\left\{\frac{d}{dt}\left(\frac{\psi\dot{\phi}}{\dot{T}}\right)-\frac{\dot{\psi}\dot{\phi}}{\dot{T}}\right\}q^2+\left(\dot{\mu}-\frac{\dot{\phi}^2}{2\dot{T}}\right)q+2Kk=0 \tag{6.17}$$

which must hold for all $q>0$. This means $K=0$, which is absurd. Thus, we see that the Kepler system, in a state of zero angular momentum, is not c-equivalent to a one-dimensional free particle.

7. CONCLUDING REMARKS

As we have seen in this paper, by means of a local coordinate transformation in configuration spacetime one can get a new Lagrangian $\hat{L}$ which describes a system with a different dynamical nature than that described by the old Lagrangian L. (Thus, for instance, one can locally transform a harmonic oscillator into a free particle.) This is so, as a general rule, only if both systems belong to the same class of c-equivalent Lagrangians.

The participation of the time variable in Lagrangian transformations plays a very meaningful role.[10] It is clear that if one keeps $T = t$, and one transforms only the generalized coordinates into $Q^j = Q^j(q)$, then the new Lagrangian $\hat{L}$ will still describe the same physical system that is described by L, although in terms of a new fixed frame of coordinates. (The change from Cartesian to spherical coordinates is a simple instance of this general feature.) On the other hand, as was already remarked in Section 4, the fact that the new coordinate frame $\{(T, Q)\}$ is "moving" relative to the old frame $\{(t, q)\}$ can produce severe changes in the dynamical *description* of the system. In this sense, the Lagrange group of Lagrangian mechanics plays the same general role as the canonical group of Hamiltonian mechanics.

Finally, as a general conclusion, we would like to remark that in this paper Lagrangian mechanics has been examined under the broad scope of the principle of general covariance.[11] A *general relativistic theory of mechanics* thus arises, which is one of the many reasons that make the Lagrange formalism so important in physics.

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[10]So-called "symmetric" Lagrangian function, which treat time as just another generalized coordinate, are well known. For a complete theory along this approach, see Johns (1989). In this paper, however, we have followed the "standard" approach to Lagrangian mechanics.

[11]It is well understood today that *any* Lagrangian theory (as, for instance, the Einstein theory of gravitation) can be formulated in a generally covariant manner. This fact was pointed out by Kretschman (1917), and concurred in by Einstein (1918). See, for instance, Tolman (1962, pp. 166–168).

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